Compensatory and equivalent variation of quasilinear utility

Consider consumer A with the following utility function:

$$U_A(X, Y) = e^X Y$$

The prices are p_X and p_Y, and the income is m.

1. **Marshallian Demand and Indirect Utility Function:** Find the Marshallian demands and the indirect utility function.

2. **Minimum Expenditure Function:** Assume that p_X, p_Y and m are such that the consumer is always located in an interior solution. Obtain the minimum expenditure function.

3. **Compensatory and Equivalent Variation:** Suppose that $p_X = 1$, $p_Y = 1$ and $m = 10$. The government establishes a tax of 1 per unit on good Y. Calculate the compensatory (CV) and equivalent (EV) variations. How are they related to consumer surplus, and what does this relationship depend on?

Solution

1. Marshallian Demand and Indirect Utility Function: The utility function of A is associated with quasi-linear preferences. It is a monotonically increasing transformation of

$$U_A(X,Y) = X + \ln Y$$

The Marshallian demands are given by the following expressions:

$$X_A^M(p_X,p_Y,m) = \begin{cases} 0 & \text{if } m < p_X \\ \frac{m}{p_X} - 1 & \text{if } m \geq p_X \end{cases}$$

$$Y_A^M(p_X,p_Y,m) = \begin{cases} \frac{m}{p_Y} & \text{if } m < p_X \\ \frac{p_X}{p_Y} & \text{if } m \geq p_X \end{cases}$$

Substituting into the utility function, it is possible to obtain the indirect utility function:

$$V_A(p_X,p_Y,m) = \begin{cases} \frac{m}{p_X} & \text{if } m < p_X \\ \\ e^{\frac{m}{p_X}-1}\frac{p_X}{p_Y} & \text{if } m \geq p_X \end{cases}$$

2. Minimum Expenditure Function: According to the statement, the Marshallian demands correspond to the interior solution. Applying duality on the appropriate section of the indirect utility function, we have:

$$U_A = e^{\frac{e_A(p_X,p_Y,U_A)}{p_X}-1}\frac{p_X}{p_Y}$$

Solving for:

$$e_A(p_X,p_Y,U_A) = p_X\left[1 + \ln\left(\frac{U_A p_Y}{p_X}\right)\right]$$

3. Compensatory and Equivalent Variation: With a tax per unit, now the price of good Y is $p_Y^1 = 2$. The equivalent variation (EV) is calculated as:

$$EV = e_A(p_X^0, p_Y^0, U_A^1) - m$$

Where:

$$U_A^1 = V(p_X^1, p_Y^1, m) = \frac{e^9}{2}$$

Substituting in the EV formula:

$$EV = 1 + \ln\left(\frac{e^9}{2\cdot 1}\right) - 10$$

Therefore, $EV = -\ln(2)$.

And if we calculate the CV:

$$CV = m - e(p_X^1, p_Y^1, u_0)$$

The utility u_0 is calculated with the indirect utility function:

$$V_A(p_X,p_Y,m) = e^{\frac{m}{p_X}-1}\frac{p_X}{p_Y} = e^{\frac{10}{1}-1}\frac{1}{1} = e^9$$

We insert this into the minimum expenditure function, but evaluated with $p_Y^1 = 2$

$$e_A(p_X, p_Y, U_A) = p_X \left[1 + \ln\left(\frac{U_A p_Y}{p_X}\right)\right] = 1[1 + ln(\frac{e^9 \cdot 2}{1})] = 1 + 9 + ln(2) = 10 + ln(2)$$

Thus, we obtain the compensatory variation:

$$CV = 10 - 10 - ln(2) = -ln(2)$$

The compensatory and equivalent variations are equal because, for quasilinear preferences, there is no income effect. As they are negative, we can affirm that the individual's welfare is reduced by the tax. Note that the demand for good Y does not depend on income, so there is no income effect on this good, as long as we are in the interior solution.